# Multifractal spectrum of the phase space related to generalized thermostatistics

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**Abstract**

We consider the set of monofractals within a multifractal related to the phase space being the support of a generalized thermostatistics. The statistical weight exponent $\tau(q)$ is shown to can be modeled by the hyperbolic tangent deformed in accordance with both Tsallis and Kaniadakis exponentials whose using allows one to describe explicitly arbitrary multifractal phase space. The spectrum function $f(d)$, determining the specific number of monofractals with reduced dimension $d$, is proved to increases monotonically from minimum value $f = -1$ at $d = 0$ to maximum $f = 1$ at $d = 1$. The number of monofractals is shown to increase with growth of the phase space volume at small dimensions $d$ and falls down in the limit $d \to 1$.

*Key words:* Multifractal; exponent of statistical weight; fractal dimension spectrum; deformed exponential.
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## 1 Introduction

Very recently, we have considered a generalized thermostatistics based on a multifractal phase space, whose dimension $D$ is not equal the number $6N$ of the degrees of freedom fixed by the number of particles $N$ [1]. In such a case, the behaviour of the complex system is determined by the dimensionless volume $\gamma = \Gamma/(2\pi\hbar)^{6N}$ of the supporting phase space where $\hbar$ is Dirac-Planck constant. According to self-similarity condition, the specific statistical weight of such a system is given by the power law function [2]

$$\varpi_q(\gamma) = \gamma^{qd} \tag{1}$$

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where $q$ is multifractal index, $d \equiv D/6N \leq 1$ is reduced fractal dimension. This function should be multiplied by the specific number of monofractals with dimension $d$

$$\mathcal{N}_d(\gamma) = \gamma^{-f(d)} \tag{2}$$

which are contained in multifractal whose spectrum is determined by a function $f(d)$. As a result, the whole statistical weight, being the multifractal measure, takes the form

$$w_q(\gamma) \equiv \int\limits_0^1 \varpi_q(\gamma)\mathcal{N}_d(\gamma)\rho(d)\mathrm{d}d = \int\limits_0^1 \gamma^{qd-f(d)}\rho(d)\mathrm{d}d \tag{3}$$

where $\rho(d)$ is a density distribution over dimensions $d$. Using the method of steepest descent, we arrive at the power law

$$w_q(\gamma) \simeq \gamma^{\tau(q)} \tag{4}$$

which generalizes the simplest relation (1) due to replacement of the bare fractal dimension $d$ related the index $q = 1$ by the multifractal function

$$\tau(q) = qd_q - f(d_q). \tag{5}$$

Here, the fractal dimension $d_q$ relates to given parameter $q$ to be defined by the conditions of application of the steepest descent method:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}d}\right|_{d=d_q} = q, \quad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}d^2}\right|_{d=d_q} < 0. \tag{6}$$

Inserting the statistical weight (4) into the deformed entropy [3,4]

$$S(W) = \int\limits_{\gamma(1)}^{\gamma(W)} \frac{\mathrm{d}\gamma}{w(\gamma)} \tag{7}$$

related to the whole statistical weight $W$ arrives at thermostatistical scheme governed by the Tsallis formalism of the non-extensive statistics [5]. Within such a scheme, the non-extensivity parameter is determined by the multifractal index (5) which monotonically increases, taking value $\tau = 0$ at $q = 1$ and $\tau \simeq 1$ at $q \to \infty$. It is appeared, the multifractal function $\tau(q)$ is reduced to the specific heat to determine, together with the inverted value $\bar{\tau}(q) \equiv 1/\tau(q) > 1$, both statistical distributions and thermodynamic functions of a complex system [1]. In this way, the entropy (7) will be positive definite, continuous, symmetric, expandable, decisive, maximal, concave, Lesche stable and fulfilling a generalized Pesin-like identity if the exponent (5) varies within the interval $[0, 1]$ [5].

In this letter, we are aimed to model possible analytical forms of both multifractal index $\tau(q)$ and related spectrum $f(d)$. In Section 2 we show that the monotonically increasing function $\tau(q)$ is presented by the hyperbolic tangent deformed in accordance with both Tsallis and Kaniadakis procedures whose using allows one to describe explicitly arbitrary multifractal phase space. Section 3 is devoted to consideration of the multifractal spectrum $f(d)$ which determines the number of monofractals within the multifractal with the specific dimension $d$. Section 4 concludes our consideration.

## 2 Analytically modeling multifractal

As the simplest case, we take the function $\tau(q)$ in the form

$$\tau = \tanh\left[\mathcal{D}(q-1)\right], \tag{8}$$

being determined by parameter $\mathcal{D} > 0$ and argument $q \in [1, \infty)$. According to Fig.1, related multifractal dimension function [2]

$$D_q = \frac{\tau(q)}{q-1} \tag{9}$$

monotonically decreases from maximum value $D_0 = \mathcal{D}$ to $D_\infty = 0$ with $q$

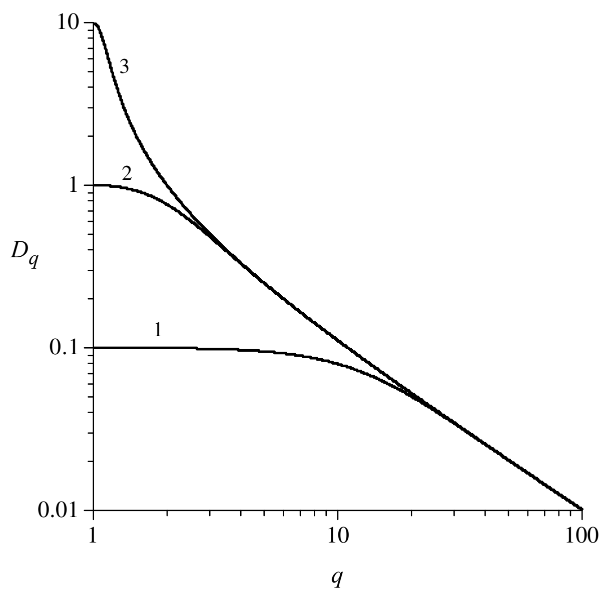


Fig. 1. Multifractal dimension function related to the dependence (8) (curves 1, 2, 3 correspond to $\mathcal{D} = 0.1, 1.0, 10$).

increase. However, maximum value of the fractal dimension $D_q$ is fixed by the magnitude $D_0 = 1$, so that one should put $\mathcal{D} = 1$ in the dependence (8). As a result, it takes quite trivial form.

As the $\tau(q)$ function increases monotonically within narrow interval $[0, 1]$, one has a scanty choice of its modeling. One of such possibilities is opened with using deformation of the hyperbolic tangent (8) at $\mathcal{D} = 1$. At present, two analytical procedures are extensively popularized. The first of these is based on the Tsallis exponential [6]

$$\exp_\kappa(x) \equiv \begin{cases} (1+\kappa x)^{1/\kappa} & \text{at} \quad 1+\kappa x > 0, \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where deformation parameter $\kappa$ takes positive values. The second procedure has been proposed by Kaniadakis [7] to determine the deformed exponential

$$\exp_\kappa(x) \equiv \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{1/\kappa}. \tag{11}$$

With using these definitions, the deformed tangent (8) takes the form

$$\tau_\kappa(q) = \tanh_\kappa(q-1) \equiv \frac{\exp_\kappa(q-1) - \exp_\kappa(1-q)}{\exp_\kappa(q-1) + \exp_\kappa(1-q)} \tag{12}$$

where the multifractal index $q$ varies within the domain $[1, \infty)$.

The $q$-dependencies of the multifractal index $\tau(q)$ and its inverted value $\bar{\tau}(q) = 1/\tau(q)$ are shown in Fig.2 at different magnitudes of both Tsallis and Kani-

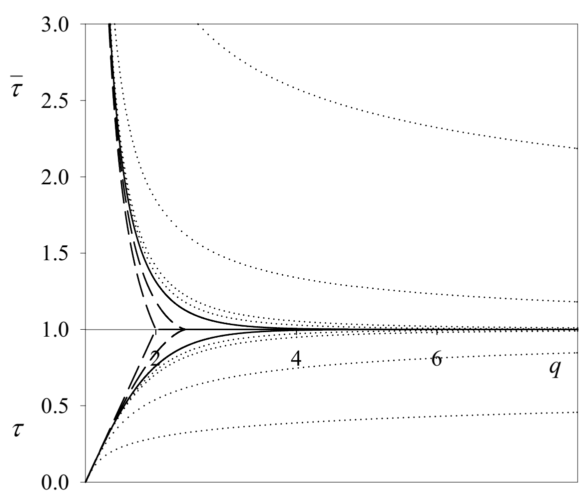


Fig. 2. The $q$-dependencies of the multifractal index $\tau$ and its inverted value $\bar{\tau} = 1/\tau$ (solid line corresponds to $\kappa = 0$, dashed curves relate to the Tsallis deformation with $\kappa = 0.7, 1$; dotted lines correspond to the Kaniadakis one at $\kappa = 0.7, 1, 3, 10$).

adakis deformation parameters $\kappa$. It is principally important, the first of these deformations arrives at more fast variations of both indices $\tau(q)$ and $\bar{\tau}(q)$ in

comparison with non-deformed hyperbolic tangent $\tau_0 = \tanh_0(q-1)$, whereas the Kaniadakis deformation slows down these variations with $\kappa$ increase.

Characteristic peculiarity of the Tsallis deformation consists in breaking dependencies $\tau(q)$, $\bar{\tau}(q)$ in the point $q_0 = (1+\kappa)/\kappa$ where the second terms in both numerator and denominator of the definition (12) take the zero value. As a result, the multifractal index (12) obtains the asymptotics

$$\tau_\kappa^{(Ts)} \simeq \begin{cases} (q-1) - \frac{1-\kappa^2}{3}(q-1)^3 & \text{at} \quad 0 < q-1 \ll 1, \\ 1 - 2(\kappa/2)^{1/\kappa}(q_0 - q)^{1/\kappa} & \text{at} \quad 0 < q_0 - q \ll q_0. \end{cases} \tag{13}$$

For $\kappa = 1$ the dependence $\tau_1^{(Ts)}(q)$ takes the simplest form: $\tau_1^{(Ts)} = q-1$ at $1 \leq q \leq 2$, and $\tau_1^{(Ts)} = 1$ at $q > 2$. It is worthwhile to notice, the Tsallis deformation parameter can not take values $\kappa > 1$ because these relate to fractal dimensions $D_q > 1$ at $q \neq 0$.

At the Kaniadakis deformation, the multifractal index $\tau_\kappa(q)$ varies smoothly to be characterized by the following asymptotics:

$$\tau_\kappa^{(K)} \simeq \begin{cases} (q-1) - \frac{2+\kappa^2}{6}(q-1)^3 & \text{at} \quad 0 < q-1 \ll 1, \\ 1 - 2\left[2\kappa(q-1)\right]^{-2/\kappa} & \text{at} \quad \kappa(q-1) \gg 1. \end{cases} \tag{14}$$

In contrast to the Tsallis case, here the deformation parameter can take arbitrary values to give the simplest dependence $\tau_1^{(K)} = \frac{q-1}{\sqrt{1+(q-1)^2}}$ at $\kappa = 1$.

As a function of the index $q$, the fractal dimension (9) falls down monotonically as shown in Fig.3a. According to Eqs.(13), at the Tsallis deformation, one has

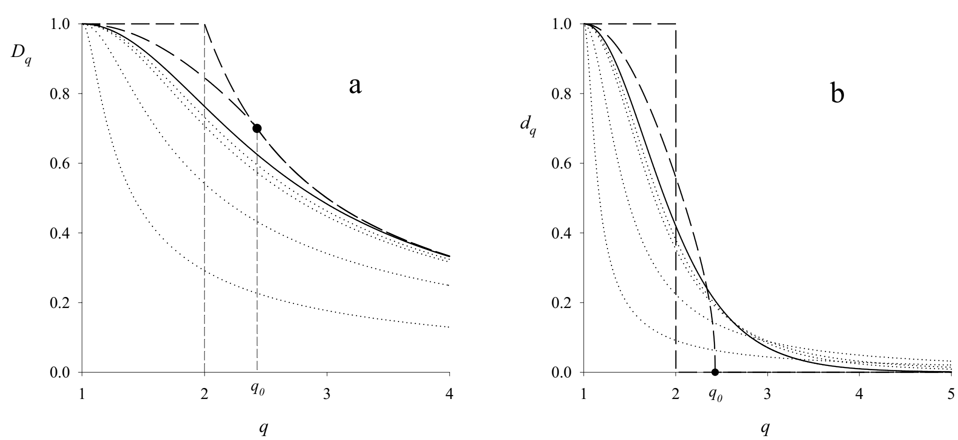


Fig. 3. Spectra of the fractal (a) and specific (b) dimensions of the phase space (the choice of the deformation parameters is the same as in Fig.2).

broken dependence $D(q)$, being characterized by the asymptotics

$$D_q^{(Ts)} \simeq \begin{cases} 1 - \frac{1-\kappa^2}{3}\left(q-1\right)^2 & \text{at} \quad 0 < q-1 \ll 1, \\ \frac{1}{q-1} - 2\left(\kappa/2\right)^{1/\kappa} \frac{(q_0-q)^{1/\kappa}}{q-1} & \text{at} \quad 0 < q_0 - q \ll q_0. \end{cases} \tag{15}$$

In the limit case $\kappa = 1$, the phase space is smooth ($D_q^{(Ts)} = 1$) within the interval $1 \leq q \leq 2$. For the Kaniadakis deformation, the fractal dimension $D_q^{(K)}$ is given by smoothly falling down curve whose slope increase with the deformation parameter growth. According to Eqs.(14), in this case, one has the asymptotics

$$D_q^{(K)} \simeq \begin{cases} 1 - \frac{2+\kappa^2}{6}\left(q-1\right)^2 & \text{at} \quad 0 < q-1 \ll 1, \\ (q-1)^{-1} - 2\left(2\kappa\right)^{-2/\kappa}\left(q-1\right)^{-(\kappa+2)/\kappa} & \text{at} \quad \kappa(q-1) \gg 1. \end{cases} \tag{16}$$

At $\kappa = 1$, the typical dependence reads $D_q^{(K)} = \left[1 + (q-1)^2\right]^{-1/2}$.

## 3 Multifractal spectrum

At given multifractal index $\tau(q)$, the spectrum function $f(d)$ is defined by inverted Legendre transformation (5) where the specific dimension reads

$$d_q = \frac{\mathrm{d}\tau}{\mathrm{d}q}. \tag{17}$$

As shows Fig.3b, the dependence $d(q)$ has monotonically falling down form to take the value $d_q = 0$ at $q > q_0 \equiv (1+\kappa)/\kappa$ for the Tsallis deformation. In this case, asymptotical behaviour is characterized by Eqs.(13), according to which one obtains

$$d_q^{(Ts)} \simeq \begin{cases} 1 - (1-\kappa^2)\left(q-1\right)^2 & \text{at} \quad 0 < q-1 \ll 1, \\ \left(\kappa/2\right)^{(1-\kappa)/\kappa}\left(q_0-q\right)^{(1-\kappa)/\kappa} & \text{at} \quad 0 < q_0 - q \ll q_0. \end{cases} \tag{18}$$

In the limit $\kappa \to 1$, the dependence $d^{(Ts)}(q)$ takes the step-like form being $d_q = 1$ within the interval $1 \leq q \leq 2$ and $d_q = 0$ otherwise.

For the Kaniadakis deformation, Eqs.(14) arrive at the asymptotics

$$d_q^{(K)} \simeq \begin{cases} 1 - \frac{2+\kappa^2}{2}(q-1)^2 & \text{at} \quad 0 < q-1 \ll 1, \\ 2^{2(\kappa-1)/\kappa}\left[\kappa(q-1)\right]^{-(2+\kappa)/\kappa} & \text{at} \quad \kappa(q-1) \gg 1. \end{cases} \tag{19}$$

The typical behaviour is presented by the dependence $d_q^{(K)} = [1+(q-1)^2]^{-3/2}$ related to $\kappa = 1$.

The multifractal spectrum is defined by the equality

$$f(d) = dq_d - \tau(q_d) \tag{20}$$

being conjugated Eq.(5). Here, the specific multifractal index $q_d$ is determined by Eq.(17) which arrives at the limit relations (18), (19). With its using, one obtains the asymptotics

$$f^{(Ts)} \simeq \begin{cases} d - \frac{2}{3}(1-\kappa^2)^{-1/2}(1-d)^{3/2} & \text{at} \quad 0 < 1-d \ll 1, \\ -\left[1 + 2(1/\kappa - 1)d^{1/(1-\kappa)}\right] + (1+1/\kappa)d & \text{at} \quad d \ll 1 \end{cases} \tag{21}$$

for the Tsallis deformation, and the relations

$$f^{(K)} \simeq \begin{cases} d - \frac{2}{3}\left(1 + \frac{\kappa^2}{2}\right)^{-1/2}(1-d)^{3/2} & \text{at} \quad 0 < 1-d \ll 1, \\ -\left[1 - 2^{(\kappa-4)/(2+\kappa)}(1+2/\kappa)d^{2/(2+\kappa)}\right] + d & \text{at} \quad d \ll 1, \end{cases} \tag{22}$$

characterizing the Kaniadakis deformation.

As shows Fig.4, for finite deformation parameters $\kappa < \infty$, a spectrum function increases monotonically, taking the minimum value $f = -1$ at $d = 0$ and maximum $f = 1$ at $d = 1$. By this, the derivative $f' \equiv \mathrm{d}f/\mathrm{d}d$ equals $f'(0) = \infty$ on the left boundary and $f'(1) = 1$ on the right one. It is characteristically, the whole set of the spectrum functions is bounded by the limit dependencies $f^{(Ts)} = 2d - 1$ and $f^{(K)} = d$, the first of which relates to limit magnitude of the Tsallis deformation parameter $\kappa = 1$ whereas the second corresponds to the Kaniadakis limit $\kappa = \infty$. Typical form of the spectrum function is presented by the dependencies

$$f^{(K)} = \begin{cases} -d\ln\left(\frac{\sqrt{d}}{1+\sqrt{1-d}}\right) + \left(d - \sqrt{1-d}\right) & \text{at} \quad \kappa = 0, \\ -\left(1 - d^{2/3}\right)^{3/2} + d & \text{at} \quad \kappa = 1. \end{cases} \tag{23}$$

It may seem, at first glance, that negative values of the spectrum function $f(d)$ has not a physical meaning. To clear up this question, let us take the set of

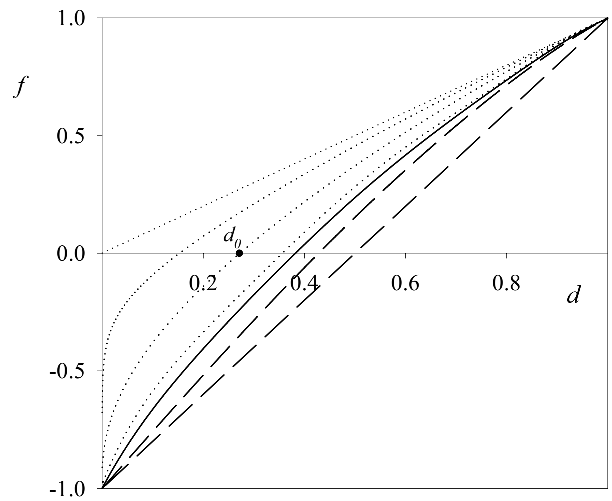


Fig. 4. Spectrum function of the multifractal phase space (solid line corresponds to $\kappa = 0$, dashed curves relate to the Tsallis deformation with $\kappa = 0.7, 1$; dotted lines correspond to the Kaniadakis one at $\kappa = 1, 3, 10, \infty$).

monofractals with the reduced dimension $d = 0$. Obviously, such monofractals relate to the whole set of the phase space points, whose number equals the dimensionless volume $\gamma$. Just such result gives the definition (2) in the point $d = 0$ where $f = -1$. On the other hand, in opposite case $d = 1$ where $f = 1$, we obtain the very thing the number of monofractals with volume $\gamma$ equals $\mathcal{N}_1 = \gamma^{-1}$ to give one multifractal of the same volume $\gamma$. In this way, a single monofractal is contained in the multifractal at condition $f(d) = 0$ which takes place at the reduced dimension $d_0$ whose dependence on the deformation parameter $\kappa$ is shown in Fig.5.

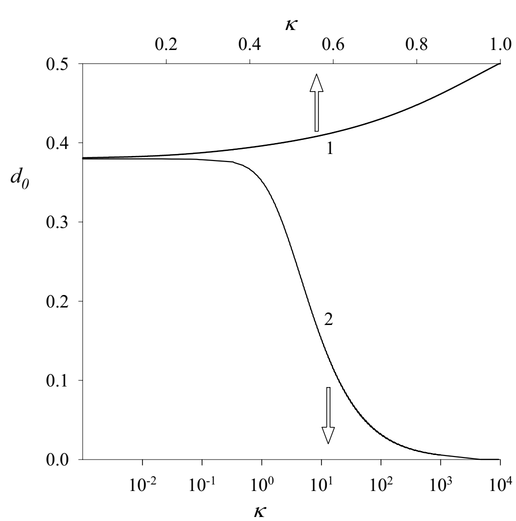


Fig. 5. Dimension $d_0$ related to the condition $f(d) = 0$ as function of the parameter $\kappa$ (curve 1 corresponds to the Tsallis deformation, curve 2 – to the Kaniadakis one; positive values $f(d)$ relate to the domain $d > d_0$).

The dependence of the number $\mathcal{N}$ of monofractals containing in the phase space volume $\gamma$ related to the multifractal with the specific dimension $d$ is shown in Fig.6. It is seen, the number $\mathcal{N}$ increases with the $\gamma$ volume growth

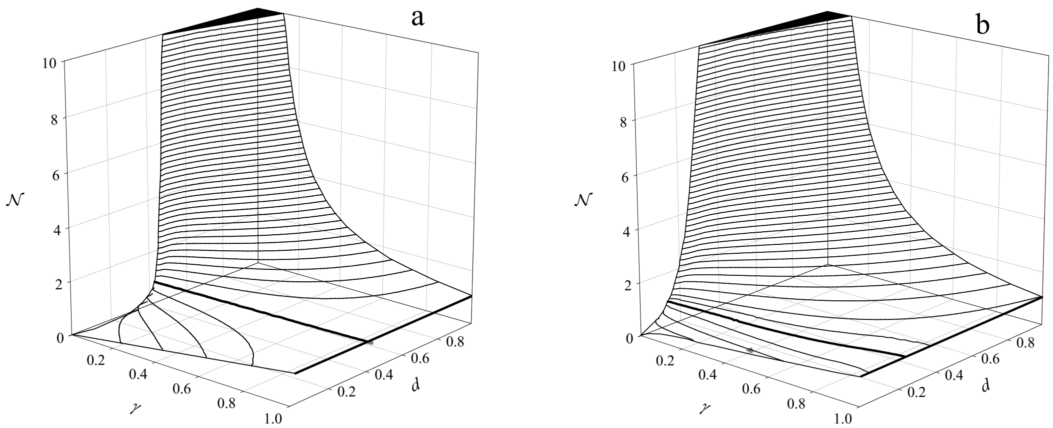


Fig. 6. The number $\mathcal{N}$ of monofractals within the phase space volume $\gamma$ related to the specific dimension $d$ at the deformation parameters $\kappa = 0$ (a) and $\kappa = 10$ (b) (different levels $\mathcal{N} = \text{const}$ are shown with thin curves, thick lines relate to $\mathcal{N} = 1$).

at small dimensions $d$, whereas in the limit $d \to 1$ the dependence $\mathcal{N}(\gamma)$ becomes falling down to give infinitely increasing numbers $\mathcal{N}$ at $\gamma \to 0$. The speed of such increase growths monotonically with both decrease of the Tsallis deformation parameter $\kappa$ and increase of the Kaniadakis one.

## 4 Conclusions

The whole set of monofractals within a multifractal related to the phase space, which gives the support of a generalized thermostatistics, is modeled by the exponent $\tau(q)$ that determines the statistical weight (4) at given volume $\gamma$. To be the entropy (7) concave, Lesche stable et cetera, the exponent $\tau(q)$ should be a function, monotonically increasing within the interval $[0, 1]$ at multifractal exponent variation within the domain $[1, \infty)$. The simplest case of such a function gives the hyperbolic tangent $\tau = \tanh(q-1)$ whose deformation (12) defined in accordance with both Tsallis and Kaniadakis exponentials (10), (11) allows one to describe explicitly arbitrary multifractal phase space. In this way, the Tsallis deformation arrives at more fast variations of the statistical weight index $\tau(q)$ in comparison with non-deformed hyperbolic tangent, whereas the Kaniadakis one slows down these variations with increasing the deformation parameter $\kappa$. All possible dependencies $\tau(q)$ are bounded from above by the linear function $\tau^{(Ts)} = q-1$ at $q \in [1, 2]$ which is transformed into the constant $\tau = 1$ at $q > 2$. This dependence relates to the smooth phase space within the Tsallis interval $q \in [1, 2]$.

The dependence (2) of the number of monofractals within the phase space volume $\gamma$ related to the multifractal with the specific dimension $d$ is determined by the spectrum function $f(d)$. This function increases monotonically, taking the minimum value $f = -1$ at $d = 0$ and maximum $f = 1$ at $d = 1$; by this, its derivative equals $f'(0) = \infty$ on the left boundary and $f'(1) = 1$ on the right one. The whole set of the spectrum functions is bounded by the limit dependencies $f^{(Ts)} = 2d - 1$ and $f^{(K)} = d$, the first of which relates to limit magnitude of the Tsallis deformation parameter $\kappa = 1$ and the second one corresponds to the Kaniadakis limit $\kappa = \infty$. The number of monofractals within the multifractal increases with the $\gamma$ volume growth at small dimensions $d$ and falls down in the limits $d \to 1$ to give infinitely increasing at $\gamma \to 0$.

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